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ON THE MODELLING OF IMPERFECT REPAIRS FOR A CONTINUOUSLY MONITORED GAMMA WEAR PROCESS THROUGH AGE REDUCTION

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Abstract

A continuously monitored system is considered, which is subject to accumulating deterioration modeled as a gamma process. The system fails when its degradation level exceeds a limit threshold. At failure, a delayed replacement is performed. To shorten the down period, a condition-based maintenance strategy is applied, with imperfect repair. Mimicking virtual age models used for recurrent events, imperfect repair actions are assumed to lower the system degradation through a first-order arithmetic reduction of age model. Under these assumptions, Markov renewal equations are obtained for several reliability indicators. Numerical examples illustrate the behavior of the system.

Keywords: imperfect maintenance; gamma process; ARA1; Markov renewal process; Markov renewal equation; delay time; maintenance policy.

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1. Introduction

Most of the systems suffer a physical degradation before the failure. A classical stochastic model to describe a non-decreasing accumulated random degradation is the gamma process. A gamma process is a stochastic process with independent, non-

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negative and gamma distributed increments with common scale parameter. This process is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, etc. [15].

For deteriorating systems, when the degradation level reaches a certain level, the system is no longer able to function satisfactorily. Since it is generally less costly to replace a system before it has failed, maintenance policies based on the system condition are usually proposed, aiming at preventing failures. It has been proved that such maintenance strategies minimize the maintenance cost, improve operational safety and reduce the quantity and severity of in-service system failures, see [2, 9, 11] e.g.. Condition-based maintenance is based on data collected online through continuous monitoring or inspections. Based on the information data, different maintenance actions are programmed. The condition of the system after a maintenance action depends on the maintenance efficiency considering two extreme cases: a minimal repair, where the condition of the system after the repair is just the same as before (As Bad As Old: ABAO), and a perfect repair, when the condition of the system after the repair is the same as if it were new (As Good As New: AGAN). Reality lies between these two extreme cases [7]. Since Chaudhuri and Sahu [6] considered the concept of imperfect maintenance, many models have been analyzed (see Pham and Wang [8] and Castro [5] for a review on imperfect maintenance models).

In the literature, several optimization models for a system subject to an accumulated degradation and under an imperfect maintenance scheme have been proposed. Newby and Baker [13], using the concept of partial repair given by Stadje and Zuckerman [17], described the maintenance process for a system whose state is described using a bivariate stochastic process. Castanier *et al.* [4] proposed a condition-based maintenance model where the effect of the imperfect maintenance is a random function of the observed deterioration of the system. Nicolai *et al.* [14] implemented different imperfect maintenance actions in systems whose degradation is modelled by a non-stationary gamma process. The effect of the maintenance action is twofold: on the one hand, to reduce the system degradation by a random amount and, on the other hand, to modify the structural parameters of the degradation process. The analysis of the model proposed by Nicolai *et al.* [14] is performed assuming that the effect of the imperfect maintenance actions annihilates the overshoot of the gamma process, whereas the present study takes it into account.

The modelling assumptions of the present paper are inspired by [2, 11], where the reader may find practical justifications for them: a system is considered, subject to a cumulative gradual random deterioration modelled as an homogeneous gamma process. A perfect and continuous monitoring controls the deterioration of the system. The system fails when its degradation level exceeds the threshold L and a signal is immediately sent to the maintenance team. They take τ units of time to arrive on site, and next perform a corrrective replacement. Compared to τ, this corrective replacement is short and it is considered as instantaneous. To reduce the system downtime, a preventive maintenance policy is proposed. Under this maintenance strategy, the signal is sent to the maintenance team as soon as the degradation level exceeds a preventive threshold M $(0 < M < L)$. It takes the same delay τ for the maintenance team to arrive, and maintenance actions are assumed to be instantaneous too. A major difference between the present study and [2, 11] is that all repairs are assumed to be perfect (AGAN) in the quoted papers. We here consider that it depends on the deterioration level at maintenance times: if the system is found failed or too degraded, a perfect corrective or preventive replacement is performed, accordingly. Otherwise, an imperfect repair is applied. Unlike most of maintenance models that combine degradation processes and imperfect maintenance actions, the maintenance effect is here modelled through a first-order Arithmetic Reduction of Age, mimicking an ARA1 model for recurrent events [7]. The maintenance efficiency is hence controlled through an Euclidian parameter ρ, allowing all situations from perfect (AGAN) to minimal (ABAO) repairs. Within such a setting, the objective of the paper is to analyze the transient behavior of the system, which is done in the framework of semi-regenerative processes with continuous space state.

The paper is structured as follows. In Section 2, the functioning of the initial system is described and the preventive maintenance policy is showed. Section 3 develops the mathematical formulation that describes the functioning of the system under the preventive maintenance policy explained in the previous section. Section 4 and 5 are focused on the calculus of different transient reliability measures, which are proved to fulfill Markov renewal equations. Section 6 shows some numerical results for these reliability measures. Note that, due to the complexity of the Markov renewal equations obtained previously, all numerical computations are here performed through Monte-Carlo simulations. Section 7 concludes.

2. Description of the system and of the maintenance strategy

As we explain before, this section describes the initial functioning of the system and the introduction of a maintenance strategy to try to improve some performance measures of the system.

2.1. The initial system

An unitary system is considered, with intrinsic deterioration modelled by a gamma process $(X_t)_{t\geq 0}$, where X_t is gamma distributed $\Gamma(\alpha t, \beta)$ with probability distribution function (p.d.f.)

$$f_t(x) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x} \mathbf{1}_{\mathbb{R}_+}(x),$$

where $\mathbf{1}_{\{\}}$ stands for the indicator function and $\alpha, \beta > 0$. The cumulative distribution function (c.d.f.) and survival function (s.f.) of X_t are denoted by F_t and $\bar{F}_t$ in the following, respectively. A gamma process also is a Lévy process, with Lévy measure given by $\mu(ds) = \alpha \frac{e^{-\beta s}}{s} \mathbf{1}_{\mathbb{R}_+^*}(s)\, ds$.

Recalling that L is the failure threshold, with $L > 0$, the time to failure of the system is the reaching time of level L :

$$\sigma_L = \inf(t > 0 : X_t > L).$$

At time σ_L, a signal is sent to the maintenance team which arrives at time $\sigma_L + \tau$ and instantaneously replaces the out-of-order system by an identical new one. The system is hence replaced by a new one at time $\sigma_L + \tau$ and the system is unavailable from σ_L up to $\sigma_L + \tau$.

2.2. The preventive maintenance policy

As we exposed in the introduction, an alert signal is preventively sent to the maintenance team as soon as the system reaches a preventive maintenance level M ($0 \leq M \leq L$), namely at time σ_M. At time $\sigma_M + \tau$, the maintenance team is ready to operate and tries to adjust the system (Preventive Maintenance action). Just as in an ARA1 model for recurrent events [7], a Preventive Maintenance (PM) action is considered to remove only some part (ρ per cent) of the age accumulated by the system since the last PM action (or since time $t = 0$), where $\rho \in (0,1)$. The PM action tends to be perfect when ρ goes to 1 (As Good As New repair) and to have no effect when ρ goes to 0 (As Bad As Old repair). In the present situation and because of possible large jumps for a gamma process ($X_{\sigma_M} \in (M, +\infty[$ almost surely), such a PM action may however be unsufficient to bring the system back to a lower level than M (details in the following). In that case and according to the previously described PM policy, a second PM action should immediately be planned, which is not coherent. We consequently consider that, in case the system deterioration level remains beyond the PM level M after a PM adjustment, the system is too deteriorated and it is preventively replaced (PR). To sum up, there consequently are three possible actions at maintenance times:

- a corrective replacement (CR) if the system is failed when the maintenance team arrives,
- a single preventive maintenance action (PM) if this PM action brings the system deterioration level below M,
- a PM action + a Preventive Replacement (PR) if the PM action does not succeed in bringing the system deterioration level below M.

All the maintenance actions are considered as instantaneous.

To specifically describe the PM policy, we shall make use of independent copies of $(X_t)_{t\geq 0}$, denoted by $\left(X_t^{(n)}\right)_{t\geq 0}$ for $n = 1, 2, ...$ Corresponding reaching times of the threshold L (resp. M) are denoted by $\sigma_L^{(n)}$ (resp. $\sigma_M^{(n)}$) for $n = 1, 2, ...$ and we set $(Y_t)_{t\geq 0}$ to be the process describing the evolution of the maintained system.

Let $U_1 = S_1 = \sigma_M^{(1)} + \tau$ be the time of the first maintenance action. We then have $Y_t = X_t^{(1)}$ for all $t < S_1$. At time S_1, different cases are possible:

- If $X_{U_1}^{(1)} > L$: the system failed before S_1. An instantaneous corrective replacement (CR) takes place at time $S_1 = \sigma_M^{(1)} + \tau$. We then set: $Y_{S_1} = 0$.
- If $X_{U_1}^{(1)} \leq L$: a PM action puts the system back to its deterioration level at time $(1-\rho)\, U_1$, which is $X_{(1-\rho)U_1}^{(1)}$.
    - if $X_{(1-\rho)U_1}^{(1)} > M$: the system is considered to be unmaintainable and it is

replaced by a new one (PR action) at time S_1, hence $Y_{S_1} = 0$.

- if $X^{(1)}_{(1-\rho)U_1} \leq M$: the system deterioration level after the PM action is $Y_{S_1} = X^{(1)}_{(1-\rho)U_1}$.

Starting from Y_{S_1} after the first maintenance action, the evolution of the system is assumed to be independent of $(Y_t)_{t<S_1}$ and is modelled by $\left(X^{(2)}_t\right)_{t\geq 0}$ up to the second maintenance action. The reaching time of level M then is

$$\inf\left(t > S_1 : Y_{S_1} + X^{(2)}_{t-S_1} > M\right) = S_1 + \sigma^{(2)}_{M-Y_{S_1}}.$$

A second maintenance action is then planned at time $S_2 = S_1 + U_2$, with $U_2 = \sigma^{(2)}_{M-Y_{S_1}} + \tau$.

More generally, assume S_1, ..., S_{n-1} and $(Y_t)_{t\leq S_{n-1}}$ to be constructed, with $n \geq 2$. Let $U_n = \sigma^{(n)}_{M-Y_{S_{n-1}}} + \tau$ and $S_n = S_{n-1} + U_n$. We first set $Y_t = Y_{S_{n-1}} + X^{(1)}_{t-S_{n-1}}$ for all $S_{n-1} < t < S_n$, and consequently: $Y_{S_n^-} = Y_{S_{n-1}} + X^{(n)}_{U_n}$ (almost surely).

- If $Y_{S_n^-} > L$: the system failed before S_n, hence $Y_{S_n} = 0$.
- If $Y_{S_n^-} \leq L$: a PM action puts the system back to the deterioration level $Y_{S_{n-1}} + X^{(n)}_{(1-\rho)U_n}$.
  - if $Y_{S_{n-1}} + X^{(n)}_{(1-\rho)U_n} > M$: the system is unmaintainable and it is replaced by a new one at time S_n, hence $Y_{S_n} = 0$,
  - if $Y_{S_{n-1}} + X^{(n)}_{(1-\rho)U_n} < M$:, the system deterioration level after the PM action is $Y_{S_n} = Y_{S_{n-1}} + X^{(n)}_{(1-\rho)U_n}$.

A new maintenance action is next planned at time $S_{n+1} = S_n + U_{n+1}$, with $U_{n+1} = \sigma^{(n+1)}_{M-Y_{S_n}} + \tau$. After a maintenance action at time S_n, the future evolution of the maintained system $(Y_t)_{t\geq S_n}$ depends on the past $(Y_t)_{t\leq S_n}$ only through Y_{S_n} and the process $(Y_t)_{t\geq 0}$ appears as a semi-regenerative process with underlying Markov renewal process $(S_n, Y_{S_n})_{n\in\mathbb{N}}$ and inter-arrival times the U_n's, see [1]. Note that the sequence $\left(S_n, \left(Y_{S_n}, Y_{S_n^-}\right)\right)_{n\in\mathbb{N}}$ also is a Markov renewal process, which will be used later on for obtaining the Markov renewal equations for both reliability and cost functions.

This age-based maintenance policy is illustrated in Figure 1: at the end of the first semi-cycle, a PM action puts the system back to $Y_{S_1} = X^{(1)}_{(1-\rho)U_1} < M$. At the end of the second semi-cycle, the system is failed and a corrective replacement leads to

$Y_{S_2} = 0$. At the end of the third semi-cycle, a PM action puts the system back to $Y_{S_2} + X^{(3)}_{(1-\rho)U_3} \geq M$ and a preventive replacement leads to $Y_{S_3} = 0$.

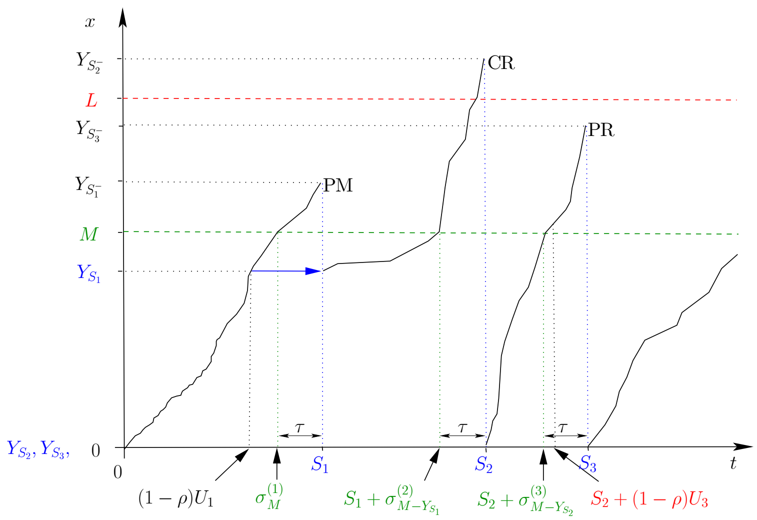

FIGURE 1: The condition-based maintenance policy

In case M goes to 0^+, the signal is immediately sent to the maintenance team after a maintenance action. The next maintenance action is hence always performed after the same delay τ. Besides, at each maintenance time the system is either failed or unmaintainable. Maintenance policy is hence reduced to periodic (corrective or preventive) replacements of the system with period τ.

If M tends to L^-, maintenance policy is reduced to perform corrective replacements actions after a delay τ.

Finally, when ρ tends to 0^+, the As Bad As Old maintenance operation leads to a system replacement and therefore leads to an As Good As New repair.

3. Markov renewal process

The aim of this section is to obtain the kernel of the Markov renewal process $\left(S_n, \left(Y_{S_n}, Y_{S_n^-}\right)\right)_{n\in\mathbb{N}}$, namely the kernel $\left(q\left(x, ds, dy, dz\right)\right)_{x\in[0,M]}$ defined by:

$$\begin{aligned} q\left(x, ds, dy, dz\right) &= \mathbb{P}\left(S_1 \in ds, Y_{S_1} \in dy, Y_{S_1^-} \in dz | Y_0 = x\right) \\ &= \mathbb{P}_x\left(S_1 \in ds, Y_{S_1} \in dy, Y_{S_1^-} \in dz\right), \end{aligned}$$

for all $x \in [0, M]$, where $\mathbb{P}_x$ stands for the conditional probability given $Y_0 = x$ (and $\mathbb{E}_x$ the conditional expectation). With this notation, we recall that:

$$\begin{aligned} &\mathbb{P}\left(S_n \in ds, Y_{S_n} \in dy, Y_{S_n^-} \in dz | \sigma\left(S_1, ..., S_{n-1}, Y_{S_1}, ..., Y_{S_{n-1}}\right)\right) \\ &= \mathbb{P}\left(S_n \in ds, Y_{S_n} \in dy, Y_{S_n^-} \in dz | Y_{S_{n-1}}\right) \\ &= q\left(Y_{S_{n-1}}, ds, dy, dz\right) \end{aligned}$$

for all $n \geq 1$, where $\sigma(A)$ stands for the σ-field generated by A, where A is any set of random variables. To obtain the kernel, firstly, we deal with the probability density function (p.d.f.) of $\left(S_1, X_{(1-\rho)S_1}, X_{S_1}\right)$.

Proposition 1. *The p.d.f. of* $\left(S_1, X_{(1-\rho)S_1}, X_{S_1}\right)$ *is* $u^M(t,u,v)$ *where:*

- *if* $\tau < t < \frac{\tau}{\rho}$ *and* $M < u < v$ *:*

$$u^M(t,u,v) = f_{\rho t}(v-u) \int_M^{+\infty} f_{\tau-\rho t}(u-w) \left(\int_{w-M}^{+\infty} f_{t-\tau}(w-s)\,\mu(ds)\right) dw, \tag{1}$$

- *if* $t > \frac{\tau}{\rho}$ *and* $u < M < v$ *:*

$$u^M(t,u,v) = f_{(1-\rho)t}(u) \int_{M-u}^{+\infty} f_{\tau}(v-u-w) \left(\int_{w-(M-u)}^{+\infty} f_{\rho t-\tau}(w-s)\,\mu(ds)\right) dw, \tag{2}$$

- $u^M(t,u,v) = 0$ *elsewhere.*

Proof. Setting φ to be any measurable and non negative function, we have to compute

$$\mathbb{E}\left[\varphi\left(S_1, X_{(1-\rho)S_1}, X_{S_1}\right)\right] = \mathbb{E}\left[\varphi\left(\sigma_M+\tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau}\right)\right].$$

We first divide this expression according to whether $(1-\rho)(\sigma_M+\tau)$ is greater or smaller than σ_M, or equivalently according to whether $(1-\rho)\tau$ is greater or smaller than $\rho\,\sigma_M$, and we write:

$$\mathbb{E}\left[\varphi\left(S_1, X_{(1-\rho)S_1}, X_{S_1}\right)\right] = I_1(\varphi) + I_2(\varphi)$$

with

$$\begin{aligned} I_1(\varphi) &= \mathbb{E}\left[\varphi\left(\sigma_M+\tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau}\right)\mathbf{1}_{\{(1-\rho)\tau>\rho\ \sigma_M\}}\right], \\ I_2(\varphi) &= \mathbb{E}\left[\varphi\left(\sigma_M+\tau, X_{(1-\rho)(\sigma_M+\tau)}, X_{\sigma_M+\tau}\right)\mathbf{1}_{\{(1-\rho)\tau<\rho\ \sigma_M\}}\right]. \end{aligned}$$

The first term is equal to:

$$I_1(\varphi) = \sum_{r\geq 0}\mathbf{1}_{\{(1-\rho)\tau>\rho\ r\}}\mathbb{E}\left[\varphi\left(r+\tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau}\right)\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\right]. \tag{3}$$

Setting $\mathcal{F}_u = \sigma(X_s, 0\leq s\leq u)$ for all $u\geq 0$, let us first note that $\{X_{r^-}\leq M<X_r\}$ belongs to $\mathcal{F}_{(1-\rho)(r+\tau)}$ for each r such that $(1-\rho)\tau>\rho\,r$ (because $(1-\rho)(r+\tau)>r$). By conditionning on $\mathcal{F}_{(1-\rho)(r+\tau)}$, writing $X_{r+\tau} = X_{(1-\rho)(r+\tau)} + \left(X_{r+\tau} - X_{(1-\rho)(r+\tau)}\right)$ and using the Markov property and the independent and homogeneous increments of $(X_t)_{t\geq 0}$, we get :

$$\begin{aligned} &\mathbb{E}\left[\varphi\left(r+\tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau}\right)\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}} g\left(X_{(1-\rho)(r+\tau)}\right)\right] \end{aligned}$$

where

$$\begin{aligned} g(x) &= \mathbb{E}\left[\varphi\left(r+\tau, x, x+X_{r+\tau}-X_{(1-\rho)(r+\tau)}\right)\right] \\ &= \mathbb{E}\left[\varphi\left(r+\tau, x, x+X_{\rho(r+\tau)}\right)\right] \\ &= \int_{\mathbb{R}_+}\varphi(r+\tau, x, x+z)\, f_{\rho(r+\tau)}(z)\, dz. \end{aligned}$$

This provides:

$$\begin{aligned} &\mathbb{E}\left[\varphi\left(r+\tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau}\right)\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\right] \\ &= \int_{\mathbb{R}_+}\mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\varphi\left(r+\tau, X_{(1-\rho)(r+\tau)}, X_{(1-\rho)(r+\tau)}+z\right)\right] f_{\rho(r+\tau)}(z)\ dz. \end{aligned} \tag{4}$$

Conditionning on $\mathcal{F}_r$ and writing $X_{(1-\rho)(r+\tau)} = X_r + \left(X_{(1-\rho)(r+\tau)} - X_r\right)$, we derive:

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\varphi\left(r+\tau, X_{(1-\rho)(r+\tau)}, X_{(1-\rho)(r+\tau)}+z\right)\right]\\
&=\int_{\mathbb{R}_+}\mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\varphi\left(r+\tau, X_r+y, X_r+y+z\right)\right] f_{(1-\rho)\tau-\rho r}\left(y\right)dy
\end{aligned}$$

in the same way, noting that $X_{(1-\rho)(r+\tau)} - X_r$ is identically distributed as $X_{(1-\rho)\tau-\rho r}$. Plugging this expression successively into (4) and next into (3), we get:

$$\begin{aligned}
I_1\left(\varphi\right) = &\iint\limits_{\mathbb{R}_+^2} dy\ dz \sum_{r\geq 0} f_{(1-\rho)\tau-\rho r}\left(y\right)\ f_{\rho(r+\tau)}\left(z\right)\\
&\times \mathbf{1}_{\{(1-\rho)\tau>\rho\ r\}}\mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_r\right\}}\varphi\left(r+\tau, X_r+y, X_r+y+z\right)\right].
\end{aligned}$$

Following arguments of Proposition 2 page 76 of [3] and setting $\Delta X_r = X_r - X_{r^-}$, we obtain:

$$\begin{aligned}
I_1\left(\varphi\right) = &\iint\limits_{\mathbb{R}_+^2} dy\ dz \sum_{r\geq 0}\mathbf{1}_{\{(1-\rho)\tau>\rho\ r\}} f_{(1-\rho)\tau-\rho r}\left(y\right)\ f_{\rho(r+\tau)}\left(z\right)\\
&\times \mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_{r^-}+\Delta X_r\right\}}\varphi\left(r+\tau, X_{r^-}+\Delta X_r+y, X_{r^-}+\Delta X_r+y+z\right)\right]\\
= &\int_0^{\frac{1-\rho}{\rho}\tau} dr \iiint\limits_{\mathbb{R}_+^3} dy\ dz\ \mu\left(ds\right)\ f_{(1-\rho)\tau-\rho r}\left(y\right)\ f_{\rho(r+\tau)}\left(z\right)\\
&\quad\times \mathbb{E}\left[\mathbf{1}_{\left\{X_{r^-}\leq M<X_{r^-}+s\right\}}\varphi\left(r+\tau, X_{r^-}+s+y, X_{r^-}+s+y+z\right)\right]
\end{aligned}$$

due to the compensation formula. Almost sure continuity of $\left(X_r\right)_{r\geq 0}$ allows to substitute X_r to X_{r^-} into the previous formula. This provides:

$$\begin{aligned}
I_1\left(\varphi\right) = &\int_0^{\frac{1-\rho}{\rho}\tau} dr \iiiint\limits_{\mathbb{R}_+^4} dx\ dy\ dz\ \mu\left(ds\right)\ \mathbf{1}_{\{x\leq M<x+s\}}\\
&\times \varphi\left(r+\tau, x+s+y, x+s+y+z\right) f_{(1-\rho)\tau-\rho r}\left(y\right)\ f_{\rho(r+\tau)}\left(z\right)\ f_r\left(x\right)
\end{aligned}$$

and next:

$$\begin{aligned}
I_1\left(\varphi\right) = &\int_\tau^{\frac{\tau}{\rho}} dt \iiiint\limits_{\mathbb{R}_+^4} dx\ du\ dv\ \mu\left(ds\right)\ \mathbf{1}_{\{w-s\leq M<w\}}\\
&\times \varphi\left(t,u,v\right) f_{\tau-\rho t}\left(u-w\right)\ f_{\rho t}\left(v-u\right)\ f_{t-\tau}\left(w-s\right)
\end{aligned}$$

setting $t = r + \tau$, $u = x + s + y$, $v = x + s + y + z$, $w = x + s$ and keeping s unchanged. This gives formula (1) for $u^M(t, u, v)$ in case $\tau < t < \frac{\tau}{\rho}$ (and $M < u < v$). As for the second term, we have:

$$I_2(\varphi) = \sum_{r \geq 0} \mathbf{1}_{\{(1-\rho)\tau < \rho r\}} \mathbb{E}\left[\varphi\left(r + \tau, X_{(1-\rho)(r+\tau)}, X_{r+\tau}\right) \mathbf{1}_{\left\{X_{r^-} \leq M < X_r\right\}}\right].$$

Conditionning on $\mathcal{F}_r$ in the expectation and writing $X_{r+\tau} = X_r + (X_{r+\tau} - X_r)$, we get:

$$\begin{aligned} I_2(\varphi) = & \int_{\mathbb{R}_+} f_\tau(z)\, dz \\ & \times \sum_{r \geq 0} \mathbf{1}_{\{(1-\rho)\tau < \rho r\}} \mathbb{E}\left[\varphi\left(r + \tau, X_{(1-\rho)(r+\tau)}, X_r + z\right) \mathbf{1}_{\left\{X_{r^-} \leq M < X_r\right\}}\right] \end{aligned}$$

because $(1 - \rho)(r + \tau) < r$. Setting $X_r = X_{r^-} + \Delta X_r$, using the compensation formula and substituting X_{r^-} by X_r in a next step, we obtain:

$$\begin{aligned} I_2(\varphi) = & \iint_{\mathbb{R}_+^2} f_\tau(z)\ dz\ \mu(ds) \int_{\frac{1-\rho}{\rho}\tau}^{+\infty} dr \\ & \times \mathbb{E}\left[\varphi\left(r + \tau, X_{(1-\rho)(r+\tau)}, X_r + s + z\right) \mathbf{1}_{\{X_r \leq M < X_r + s\}}\right]. \end{aligned}$$

Conditionning on $\mathcal{F}_{(1-\rho)(r+\tau)}$, writing $X_r = X_{(1-\rho)(r+\tau)} + \left(X_r - X_{(1-\rho)(r+\tau)}\right)$ and using the fact that $X_r - X_{(1-\rho)(r+\tau)}$ is identically distributed as $X_{\rho r - (1-\rho)\tau}$, we get:

$$\begin{aligned} I_2(\varphi) = & \iiiint_{\mathbb{R}_+^4} f_\tau(z)\ dz\ \mu(ds)\ du\ dy \int_{\frac{1-\rho}{\rho}\tau}^{+\infty} dr \\ & \times \varphi(r + \tau, u, u + y + s + z) \mathbf{1}_{\{u + y \leq M < u + y + s\}} f_{(1-\rho)(r+\tau)}(u) f_{\rho r - (1-\rho)\tau}(y) \\ = & \iiiint_{\mathbb{R}_+^4} f_\tau(v - u - w)\ dw\ \mu(ds)\ du\ dv \int_{\frac{\tau}{\rho}}^{+\infty} dt \\ & \times \varphi(t, u, v) \mathbf{1}_{\{w - s \leq M - u < w\}} f_{(1-\rho)t}(u) f_{\rho t - \tau}(w - s) \end{aligned}$$

with $t = r + \tau$, $v = u + y + s + z$, $w = y + s$ and (u, s) unchanged. This provides formula (2) for $u^M(t, u, v)$ in case $t > \frac{\tau}{\rho}$ (and $u < M < v$).

Remark 1. Using the fact that the p.d.f. of (σ_M, X_{σ_M}) is

$$f_{(\sigma_M, X_{\sigma_M})}(u, y) = \int_{y-M}^{+\infty} f_u(y - s)\, \mu(ds) \tag{5}$$

for all $y > M$ and all $u > 0$ (see [3]), the function $u^M(t,u,v)$ may be written as

$$u^M(t,u,v) = f_{\rho t}(v-u) \int_M^{+\infty} f_{\tau-\rho t}(u-w)\, f_{\left(\sigma_M, X_{\sigma_M}\right)}(t-\tau, w)\, dw$$

if $\tau < t < \frac{\tau}{\rho}$ and $M < u < v$. This corresponds to some kind of intuitive result: roughly speaking, at time $\sigma_M = t-\tau$, the process $(X_r)_{r\geq 0}$ reaches level $w > M$. Next, on the time interval $(t-\tau, (1-\rho)t]$ with length $\tau - \rho t$, the level is increased of $u - w$ units and the process reaches level u at time $(1-\rho)t$. Finally, on the time interval $((1-\rho)t, t]$ with length ρt, the level is increased of $v-u$ units and the process reaches level v at time t. In case $t > \frac{\tau}{\rho}$ and $u < M < v$, we get

$$u^M(t,u,v) = f_{(1-\rho)t}(u) \int_{M-u}^{+\infty} f_\tau(v-u-w)\, f_{\left(\sigma_{M-u}, X_{\sigma_{M-u}}\right)}(\rho t - \tau, w)\ dw$$

which may be interpreted in the same way: on the interval $(0, (1-\rho)t]$, the level is increased of u units (with $u < M$). Next, starting from level u, it takes $\rho t - \tau$ time units for the process to exceed level $M - u$ with a level increment of w units in the meantime (and $w > M - u$). At time $(1-\rho)t + \rho t - \tau = t - \tau$, the level hence is $u + w$. Finally, on the time interval $(t-\tau, t]$ with length τ, the level is increased of $v - u - w$ units and the process reaches level v at time t.

We are now able to provide the kernel of the Markov renewal process $\left(S_n, \left(Y_{S_n}, Y_{S_n^-}\right)\right)_{n\in\mathbb{N}}$.

Theorem 1. *The kernel* $(q(x,ds,dy,dz))_{x\in[0,M]}$ *of the Markov renewal process* $\left(S_n, \left(Y_{S_n}, Y_{S_n^-}\right)\right)_{n\in\mathbb{N}}$ *is provided by*

$$\begin{aligned} q(x,ds,dy,dz) = {} & \mathbf{1}_{\{s>\tau\}} \mathbf{1}_{\{y\leq M<z\leq L\}}\ u^{M-x}(s, y-x, z-x)\ dy\ dz\ ds \\ & + \mathbf{1}_{\{s>\tau\}} q_x(s,z)\ \delta_0(dy)\ dz\ ds \end{aligned} \tag{6}$$

for all $x \in [0,M]$*, where* u^M *is provided by Proposition* 1 *and where*

$$\begin{aligned} q_x(s,z) = {} & \mathbf{1}_{\{L<z\}} \left(\int_0^{z-x} u^{M-x}(s,w,z-x)\ dw \right) \\ & + \mathbf{1}_{\{M<z\leq L\}} \left(\int_{M-x}^{z-x} u^{M-x}(s,w,z-x)\ dw \right). \end{aligned} \tag{7}$$

The first term in the right hand of (6) *stands for the PM case, and the two terms in the right hand of* (7) *for the CR and PR cases, respectively.*

Proof. Given that $Y_0 = x$, we set $S^x = S_1 = \tau + \sigma_{M-x}$. This provides $Y_{S_1^-} = x + X_{S^x}$ and

$$Y_{S_1} = \begin{cases} 0 \text{ if } X_{S^x} > L - x \\ 0 \text{ if } X_{S^x} \leq L - x \text{ and } X_{(1-\rho)S^x} > M - x \\ x + X_{(1-\rho)S^x} \text{ if } X_{S^x} \leq L - x \text{ and } X_{(1-\rho)S^x} \leq M - x. \end{cases}$$

For all φ measurable and non negative, we hence have:

$$\mathbb{E}_x \left(\varphi \left(S_1, Y_{S_1}, Y_{S_1^-} \right) \right) = J_1 (x) + J_2 (x)$$

with

$$J_1 (x) = \mathbb{E} \left[\varphi \left(S^x, x + X_{(1-\rho)S^x}, x + X_{S^x} \right) \mathbf{1}_{\left\{ X_{S^x} \leq L-x, X_{(1-\rho)S^x} \leq M-x \right\}} \right],$$
$$J_2 (x) = \mathbb{E} \left[\varphi \left(S^x, 0, x + X_{S^x} \right) \left(\mathbf{1}_{\{X_{S^x} > L-x\}} + \mathbf{1}_{\left\{ X_{S^x} \leq L-x, M-x < X_{(1-\rho)S^x} \right\}} \right) \right].$$

Using Proposition 1 with M substituted by $M - x$, we derive:

$$\begin{aligned} J_1 (x) &= \iiint_{\mathbb{R}_+^3} \varphi (s, x+u, x+v) \, u^{M-x} (s, u, v) \, \mathbf{1}_{\{v \leq L-x, u \leq M-x\}} \, du \, dv \, ds \\ &= \iiint_{\mathbb{R}_+^3} \varphi (s, y, z) \, \mathbf{1}_{\{y \leq M < z \leq L\}} u^{M-x} (s, y-x, z-x) \, dy \, dz \, ds \end{aligned}$$

where $y = x + u$, $z = x + v$, and

$$\begin{aligned} J_2 (x) &= \iiint_{\mathbb{R}_+^3} \varphi (s, 0, x+v) \, u^{M-x} (s, u, v) \left(\mathbf{1}_{\{L-x<v\}} + \mathbf{1}_{\{v \leq L-x, M-x<u\}} \right) \, du \, dv \, ds \\ &= \iint_{\mathbb{R}_+^2} \varphi (s, 0, z) \, \mathbf{1}_{\{L<z\}} \left(\int_x^z u^{M-x} (s, y-x, z-x) \, dy \right) \, dz \, ds \\ &+ \iint_{\mathbb{R}_+^2} \varphi (s, 0, z) \, \mathbf{1}_{\{M<z \leq L\}} \left(\int_M^z u^{M-x} (s, y-x, z-x) \, dy \right) \, dz \, ds, \end{aligned}$$

which provides the result.

We finally derive the kernel of the Markov renewal process $(S_n, Y_{S_n})_{n \in \mathbb{N}}$.

Corollary 1. *The kernel* $(\bar{q}(x, ds, dy))_{x \in [0,M]}$ *of the Markov renewal process* $(S_n, Y_{S_n})_{n \in \mathbb{N}}$ *is provided by*

$$\bar{q}(x, ds, dy) = \mathbf{1}_{\{s>\tau\}} ds \ \left\{ \mathbf{1}_{\{y \leq M\}} \ H_x (s, y) \ dy + \delta_0 (dy) \ \ (I_x (s) + D_x (s)) \right\}$$

for all $x \in [0, M]$, where

$$H_x(s,y) = \int_{M-x}^{L-x} u^{M-x}(s, y-x, v)\ dv \qquad \text{(PM case)}, \tag{8}$$

$$D_x(s) = \int_{L-x}^{+\infty} dz \left(\int_0^z u^{M-x}(s, w, z)\ dw \right) \qquad \text{(CR case)}, \tag{9}$$

$$I_x(s) = \int_{M-x}^{L-x} dz \left(\int_{M-x}^z u^{M-x}(s, y, z)\ dy \right) \qquad \text{(PR case)}. \tag{10}$$

Proof. We have:

$$\begin{aligned} \bar{q}(x, ds, dy) &= \int q(x, ds, dy, dz) \\ &= \mathbf{1}_{\{s>\tau\}} \mathbf{1}_{\{y \le M\}} ds\ dy \int_M^L u^{M-x}(s, y-x, z-x)\ dz + \mathbf{1}_{\{s>\tau\}}\ \delta_0(dy)\ ds\ \int_M^{+\infty} q_x(s,z)\, dz \\ &= \mathbf{1}_{\{s>\tau\}} ds \left\{ \mathbf{1}_{\{y \le M\}} H_x(s,y)\ dy + \delta_0(dy)\ \int_M^{+\infty} q_x(s,z)\, dz \right\} \end{aligned}$$

with

$$\begin{aligned} \int_M^{+\infty} q_x(s,z)\, dz &= \int_M^L dz \left(\int_M^z u^{M-x}(s, w-x, z-x)\ dw \right) + \int_L^{+\infty} dz \left(\int_x^z u^{M-x}(s, w-x, z-x)\ dw \right) \\ &= I_x(s) + D_x(s) \end{aligned}$$

and the result holds.

4. The reliability and availability functions

Let $R_x(t)$ be the reliability function of the maintained system at time t, namely the conditional probability that the system has been functionning from time $t = 0$ up to time t without any interruption given that it started from $Y_0 = x$ with $x \in [0, M]$:

$$R_x(t) = \mathbb{P}_x(T > t)$$

where T is the time to failure of the maintained system and $t \in \mathbb{R}_+$.

As $S_1 = \tau + \sigma_{M-x} > \tau$, let us first remark that, if $t \le \tau$, then $t < S_1$ and there is no maintenance action on $[0, t]$. In that case, $Y_u = X_u$ on $[0, t]$ and we simply get:

$$R_x(t) = \mathbb{P}(\sigma_{L-x} > t) = \mathbb{P}(X_t < L - x) = F_t(L - x)$$

for all $t \le \tau$. We next envision the case where $t > \tau$.

Theorem 2. *The reliability function fulfills the following Markov renewal equation:*

$$\begin{aligned} R_x(t) &= G_x(t) + \int_\tau^t \int_0^M R_y(t-s)\, H_x(s,y)\ ds\ dy + \int_\tau^t R_0(t-s)\, I_x(s)\ ds \qquad (11) \\ &= G_x(t) + \int_\tau^t \int_0^M R_y(t-s)\, \nu_x(ds,dy) \end{aligned}$$

for all $t > \tau$, $x \in [0, M]$, *where*

$$G_x(t) = \int_0^{M-x} f_{t-\tau}(y)\, F_\tau(L-x-y)\, dy, \qquad (12)$$

for all $t > \tau$, $x \in [0, M]$ *and*

$$\nu_x(ds, dy) = [H_x(s,y)\ dy + I_x(s)\, \delta_0(dy)]\, ds. \qquad (13)$$

with H_x *and* I_x *as in* $(8, 10)$.

Proof. Let $t > \tau$. We have:

$$R_x(t) = \mathbb{P}_x(T > t, S_1 > t) + \mathbb{P}_x(T > t, S_1 \le t) \qquad (14)$$

with

$$\begin{aligned} \mathbb{P}_x(T > t, S_1 > t) &= \mathbb{P}(X_t \le L - x, \tau + \sigma_{M-x} > t) \\ &= \mathbb{P}(X_t \le L - x, X_{t-\tau} \le M - x) \\ &= \int_0^{M-x} f_{t-\tau}(y)\, F_\tau(L-x-y)\, dy \\ &= G_x(t), \qquad (15) \end{aligned}$$

by conditioning with respect of $X_{t-\tau}$. We also have:

$$\begin{aligned} \mathbb{P}_x(T > t, S_1 \le t) &= \mathbb{E}_x\left[\mathbf{1}_{\{S_1 \le t\}} \mathbf{1}_{\{T > S_1\}} \mathbb{E}_x\left(\mathbf{1}_{\{T > t\}} | \sigma(Y_t, t \le S_1)\right)\right] \\ &= \mathbb{E}_x\left[\mathbf{1}_{\{S_1 \le t\}} \mathbf{1}_{\left\{Y_{S_1}^- \le L\right\}} R_{Y_{S_1}}(t - S_1)\right] \\ &= \iint_{[\tau,t]\times[0,M]} R_y(t-s) \int_{z=0}^{z=L} q(x, ds, dy, dz) \qquad (16) \end{aligned}$$

where $q(x, ds, dy, dz)$ is given in Theorem 1. Besides, using similar arguments as for the proof of Corollary 1, we get:

$$\int_{z=0}^{z=L} q(x, ds, dy, dz) = \left\{\mathbf{1}_{\{y \le M\}}\ H_x(s,y)\ dy + \delta_0(dy)\ I_x(s)\right\} ds, \qquad (17)$$

which provides the result by plugging (17) into (16), and next $(15 - 16)$ into (14).

We now deal with the availability function of the maintained system at time t, namely with the conditional probability that the system is working at time t given that it started from $Y_0 = x$, with $x \in [0, M]$:

$$A_x(t) = \mathbb{P}_x(Y_t < L).$$

In case $t \leq \tau \, (\leq S_1)$, both reliability and availability functions coincide:

$$A_x\,(t) = R_x\,(t) = F_t\,(L - x)$$

for all $t \leq \tau$. In case $t > \tau$, we may write

$$A_x(t) = \mathbb{P}_x(Y_t < L, S_1 > t) + \mathbb{E}_x\left[1_{\{S_1 \leq t\}} A_{Y_{S_1}}(t - S_1)\right]$$

in a similar way to Theorem 2, which provides the following Corollary.

Corollary 2. *The availability function fulfills the following Markov renewal equation*

$$A_x(t) = G_x(t) + \int_\tau^t \int_0^M A_y(t-s)\bar{q}(x, ds, dy),$$

for all $t > \tau$, all $x \in [0, M]$, where $G_x(t)$ is given by (12) and $\bar{q}$ by Corollary 1.

5. The expected cost function

Let $c_x\,(t)$ be the mean cumulated cost on $]0, t]$ given that $Y_0 = x$ with $x \in [0, M]$, that is,

$$c_x(t) = \mathbb{E}_x\left[C\left(]0, t]\right)\right].$$

where $C\,(]0, t])$ denotes the maintenance cost in $]0, t]$. We calculate $c_x(t)$ taking into account the following costs for the different maintenance actions: c_{CR} corrective replacement cost, c_{PR} preventive replacement cost, c_{PM} preventive maintenance cost and c_d downtime cost per unit time.

For $t \leq \tau$, using again $Y_u = X_u$ in $[0, t]$, we get:

$$\begin{aligned} c_x\,(t) &= c_d \mathbb{E}\left[(t - \sigma_{L-x})^+\right] \\ &= c_d \int_0^t \mathbb{P}\left(t - u > \sigma_{L-x}\right)\,du \\ &= c_d \int_0^t \bar{F}_{t-u}\,(L - x)\,du, \end{aligned}$$

where $(t-\sigma_{L-x})^+ = \max(t-\sigma_{L-x},0)$ stands for the (possible) down-time on $[0,t]$.

We next envision the case where $t>\tau$ and we consider

$$c_x(t) = \mathbb{E}_x\left[C(]0,t])\,\mathbf{1}_{\{S_1>t\}}\right] + \mathbb{E}_x\left[C(]0,t])\,\mathbf{1}_{\{S_1\le t\}}\right]. \tag{18}$$

The first term in (18) is dealt with in the following lemma.

Lemma 1. *For $t>\tau$, we have:*

$$\mathbb{E}_x\left[C(]0,t])\,\mathbf{1}_{\{S_1>t\}}\right] = c_d\ K_x(t)$$

for all $x\in[0,M]$, with

$$K_x(t) = \int_0^\tau \alpha(t-\tau, t-u, M-x, L-x)\,du,\ \ all\ x\in[0,M], \tag{19}$$

$$\alpha(t_1,t_2,M,L) = \int_0^M f_{\min(t_1,t_2)}(z)\,\bar{F}_{|t_1-t_2|}(L-z)\,dz,\ \ all\ t_1,t_2\ge 0. \tag{20}$$

Proof. Using a similar method as in Proposition 3 of [12], we have:

$$\begin{aligned}\mathbb{E}_x\left[C(]0,t])\,\mathbf{1}_{\{S_1>t\}}\right] &= c_d\mathbb{E}\left[(t-\sigma_{L-x})^+\,\mathbf{1}_{\{\sigma_{M-x}+\tau>t\}}\right]\\ &= c_d\mathbb{E}\left[\mathbf{1}_{\{\sigma_{M-x}+\tau>t\}}\int_0^{+\infty}\mathbf{1}_{\left\{(t-\sigma_{L-x})^+>u\right\}}du\right]\\ &= c_d\int_0^\tau \mathbb{P}\left[\sigma_{M-x}>t-\tau, t-u>\sigma_{L-x}\right]du\end{aligned}$$

because $\sigma_{M-x}>t-\tau$ and $t-u>\sigma_{L-x}$ imply $u\le\tau$. Now, for $u\le\tau$, by conditionning with respect of $\sigma(X_s, s\le t-\tau)$, we get:

$$\begin{aligned}\mathbb{P}\left[\sigma_{M-x}>t-\tau, t-u>\sigma_{L-x}\right] &= \mathbb{P}\left[X_{t-\tau}<M-x, X_{t-u}\ge L-x\right]\\ &= \int_0^{M-x} f_{t-\tau}(y)\,\bar{F}_{\tau-u}(L-x-y)\,dy\\ &= \alpha(t-\tau, t-u, M-x, L-x)\end{aligned}$$

which provides the result.

For the calculus of the second part of (18), we shall need the following technical lemma.

Lemma 2. *For fixed $t>\tau$, in case $S_1\le t$, the conditional expected downtime given $Y_0=x$ on the first semi-cycle is*

$$\mathbb{E}_x\left[(S_1-\sigma_L)^+\,\mathbf{1}_{\{S_1\le t\}}\right] = W_x(t)$$

for all $x \in [0, M]$, where $W_x(t)$ is given by

$$W_x(t) = W_{1,x}(t)\mathbf{1}_{\{t<2\tau\}} + W_{2,x}(t)\mathbf{1}_{\{t\geq 2\tau\}},$$

for $t > \tau$ and $x \in [0, M]$, with

$$\begin{aligned} W_{1,x}(t) &= \int_0^{t-\tau} \bar{F}_v(L-x)\ dv + \int_{t-\tau}^{\tau} \beta(v, t-\tau, M-x, L-x)\ dv \\ &\quad + \int_\tau^t \int_0^{M-x} f_{v-\tau}(y)\beta(t-v, \tau, M-x-y, L-x-y)\ dy\ dv, \\ W_{2,x}(t) &= \int_0^{\tau} \bar{F}_v(L-x)\ dv + \int_\tau^{t-\tau} \alpha\left(v-\tau, v, M-x, L-x\right)\ dv \\ &\quad + \int_{t-\tau}^t dv \int_0^{M-x} f_{v-\tau}(y)\ \beta(\tau, t-v, M-x-y, L-x-y)\ dy, \end{aligned}$$

where function $\beta\left(t_1, t_2, M, L\right)$ is given by

$$\beta\left(t_1, t_2, M, L\right) = \int_M^{+\infty} f_{\min(t_1,t_2)}\left(z\right)\bar{F}_{|t_1-t_2|}\left(L-z\right)dz, \tag{21}$$

for all $t_1, t_2 \geq 0$ and $\alpha\left(t_1, t_2, M, L\right)$ is provided in (20).

Proof. We have:

$$\begin{aligned} \mathbb{E}_x\left[\left(S_1-\sigma_L\right)^+\mathbf{1}_{\{S_1\leq t\}}\right] &= \mathbb{E}\left[\left(\sigma_{M-x}+\tau-\sigma_{L-x}\right)^+\mathbf{1}_{\{\sigma_{M-x}+\tau\leq t\}}\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}}\mathbf{1}_{\{0<u<\sigma_{M-x}+\tau-\sigma_{L-x}\}}\mathbf{1}_{\{\sigma_{M-x}+\tau\leq t\}}du\right] \\ &= \mathbb{E}\left[\int_0^{+\infty}\mathbf{1}_{\{\sigma_{L-x}<v<\sigma_{M-x}+\tau\}}\mathbf{1}_{\{\sigma_{M-x}+\tau\leq t\}}dv\right] \\ &= \int_0^t \lambda\left(v,t,\tau\right)dv \end{aligned}$$

setting $v = \sigma_{M-x} + \tau - u$ and

$$\begin{aligned} \lambda\left(v,t,\tau\right) &= \mathbb{P}\left[\sigma_{L-x} < v, v-\tau < \sigma_{M-x} \leq t-\tau\right] \\ &= \mathbb{P}\left(L-x < X_v, X_{(v-\tau)^+} \leq M-x < X_{t-\tau}\right) \end{aligned}$$

for all $0 \leq v \leq t$ and all $t > \tau$. We next have to envision different cases to compute $\lambda\left(v,t,\tau\right)$ according to the respective ordering of v and $t-\tau$, and of v and τ. Firstly, if $t-\tau < \tau$ then $t < 2\tau$. For $t < 2\tau$ if $v \leq \tau$, we consider the cases $v < t-\tau$ and $v \geq t-\tau$. If $v < t-\tau$, then

$$\lambda(v,t,\tau) = \mathbb{P}(L-x < X_v) = \bar{F}_v(L-x).$$

And, if $v \geq t - \tau$, we have

$$\begin{aligned}\lambda(v,t,\tau) &= \mathbb{P}(L - x < X_v, M - x < X_{t-\tau}) \\ &= \int_{M-x}^{\infty} f_{t-\tau}(y)\bar{F}_{v-(t-\tau)}(L - x - y)\ dy \\ &= \beta(t-\tau, v, M-x, L-x),\end{aligned}$$

where $\beta(t_1, t_2, M, L)$ is given by (21). For $v > \tau$

$$\begin{aligned}\lambda(v,t,\tau) &= \mathbb{P}(L - x < X_v, X_{v-\tau} < M - x < X_{t-\tau}) \\ &= \int_0^{M-x} f_{v-\tau}(y) \int_{M-x-y}^{\infty} f_{t-v}(w)\bar{F}_{\tau-(t-v)}(L - x - y - w)\ dy\ dw \\ &= \int_0^{M-x} f_{v-\tau}(y)\beta(t-v, \tau, M-x-y, L-x-y)\ dy.\end{aligned}$$

Hence

$$\begin{aligned}W_x(t) = &\int_0^{t-\tau} \bar{F}_v(L-x)\ dv + \int_{t-\tau}^{\tau} \beta(v, t-\tau, M-x, L-x)\ dv \\ &+ \int_{\tau}^{t}\int_0^{M-x} f_{v-\tau}(y)\beta(t-v, \tau, M-x-y, L-x-y)\ dy\ dv,\end{aligned}$$

for $t < 2\tau$ and $x \in [0, M]$. For $t > 2\tau$, we have

$$\lambda(v,t,\tau) = \mathbb{P}(L - x < X_v) = \bar{F}_v(L-x)$$

for $v < \tau$. For $v \geq \tau$, we consider two cases, that is, $v < t - \tau$ and $v \geq t - \tau$. For $v < t - \tau$, we have

$$\begin{aligned}\lambda(v,t,\tau) &= \mathbb{P}(L - x < X_v, X_{v-\tau} \leq M - x) \\ &= \int_0^{M-x} f_{v-\tau}(y)\bar{F}_{\tau}(L - x - y)\ dy \\ &= \alpha\left(v-\tau, v, M-x, L-x\right).\end{aligned}$$

Finally, for $v \geq t - \tau$, we have

$$\begin{aligned}\lambda(v,t,\tau) &= \mathbb{P}(L - x < X_v, X_{v-\tau} \leq M - x < X_{t-\tau}) \\ &= \int_0^{M-x} f_{v-\tau}(y)\ dy \int_{M-x-y}^{\infty} f_{t-v}(w)\bar{F}_{\tau-(t-v)}(L - x - y - w)\ dw \\ &= \int_0^{M-x} f_{v-\tau}(y)\beta(t-v, \tau, M-x-y, L-x-y)\ dy.\end{aligned}$$

This provides the result for $t > 2\tau$ and ends the proof.

With the previous lemmas, the following result holds.

Theorem 3. *The expected cost function at time t with $Y_0 = x$ fulfills the following Markov renewal equation*

$$c_x(t) = B_x(t) + \int_\tau^t \int_0^M c_y(t-s)\ \bar{q}\,(x, ds, dy)\,,$$

with $x \in [0, M]$, where $B_x(t)$ is given by

$$B_x(t) = c_d\left[K_x\left(t\right) + W_x(t)\right] + c_{CR} Z_x(t) + \left(c_{PR} + c_{PM}\right) Q_x(t) + c_{PM} J_x(t),$$

with $K_x\left(t\right)$ and $W_x(t)$ provided in Lemmas 1 and 2, and

$$\begin{aligned} Z_x\left(t\right) &= \int_\tau^t D_x\left(s\right) ds, \\ Q_x(t) &= \int_\tau^t I_x\left(s\right) ds, \\ J_x(t) &= \int_\tau^t \int_0^M H_x\left(s, y\right) ds\ dy \end{aligned}$$

where $H_x\left(s, y\right)$, $D_x\left(s\right)$, $I_x\left(s\right)$ are defined in $(8-10)$.

Proof. Starting from (18), we have that

$$c_x(t) = \mathbb{E}_x\left[C\left(]0, t]\right)\mathbf{1}_{\{S_1 > t\}}\right] + \mathbb{E}_x\left[C\left(]0, S_1]\right)\mathbf{1}_{\{S_1 \leq t\}}\right] + \mathbb{E}_x\left[C\left(]S_1, t]\right)\mathbf{1}_{\{S_1 \leq t\}}\right] \quad (22)$$

where the first right-hand term has been computed in Lemma 1. The second term is:

$$\mathbb{E}_x\left[C\left(]0, S_1]\right)\mathbf{1}_{\{S_1 \leq t\}}\right] = c_d W_x(t) + c_{CR} Z_x(t) + (c_{PM} + c_{PR}) Q_x(t) + c_{PM} J_x(t)$$

where $W_x\left(t\right)$ is provided in Lemma 2 and where:

$$\begin{aligned} Z_x(t) &= \mathbb{P}_x\left(S_1 \leq t, Y_{S_1^-} > L\right), \\ Q_x(t) &= \mathbb{E}_x\left[\mathbf{1}_{\{S_1 \leq t\}}\mathbf{1}_{\left\{Y_{S_1^-} \leq L\right\}}\mathbf{1}_{\left\{Y_{S_1} > M\right\}}\right], \\ J_x(t) &= \mathbb{E}_x\left[\mathbf{1}_{\{S_1 \leq t\}}\mathbf{1}_{\left\{Y_{S_1^-} \leq L\right\}}\mathbf{1}_{\left\{Y_{S_1} \leq M\right\}}\right]. \end{aligned}$$

Due to Corollary 1, we get:

$$Z_x(t) = \iiint\limits_{[\tau,t]\times[0,M]\times[L,+\infty[} q(x, ds, dy, dz) = \int_\tau^t D_x(s)\, ds,$$

$$Q_x(t) = \iiint\limits_{[\tau,t]\times[M,+\infty[\times[0,L]} q(x, ds, dy, dz) = \int_\tau^t I_x(s)\, ds,$$

$$J_x(t) = \iiint\limits_{[\tau,t]\times[0,M]\times[0,L]} q(x, ds, dy, dz) = \int_\tau^t \int_0^M H_x(s, y)\, ds\ dy.$$

As for last term of (22), by conditionning on $\sigma(Y_{S_1}, S_1)$, we have:

$$\begin{aligned}\mathbb{E}_x\left[C(]S_1, t])\mathbf{1}_{\{S_1\le t\}}\right] &= \mathbb{E}_x\left[c_{Y_{S_1}}(t - S_1)\mathbf{1}_{\{S_1\le t\}}\right] \\ &= \iint\limits_{\mathbb{R}_+^2} c_y(t - s)\mathbf{1}_{\{s\le t\}}\bar{q}(x, ds, dy),\end{aligned}$$

which ends this proof.

6. Numerical examples

In order to illustrate the analytical results, several numerical examples are here considered. To make the numerical assesments, a possibility might have been to follow [10] and use some integration scheme for integral equations with singular kernels [16] e.g., for solving the Markov renewal equations developed in the paper. Unfortunately, due to the complexity of our Markov kernel, this has not been possible. That is why the numerical computations have finally been performed through Monte-Carlo (MC) simulations. To shorten the large computing times induced by our intricate model, the parallel computer EMPIRE of the University of Extremadura has been used.

For each of the following examples, the parameters of the gamma process measuring the system intrinsic deterioration are $\alpha = 1.5$ and $\beta = 3$. The system is assumed to be new at time 0, that is $Y_0 = 0$. The failure threshold is $L = 10$. The induced approximated expected time to exceed level 10 is $\mathbb{E}(\sigma_L) \simeq 20.37$ time units. The maintenance efficiency is provided by $\rho = 0.5$. The costs associated with the different maintenance actions are $c_{CR} = 100$ monetary units (m.u.), $c_{PR} = 60$ m.u., $c_{PM} = 5$ m.u. and $c_u = 2$ m.u. per time unit. As for the delay time τ, different values are envisionned in the following.

In the first case, we take $\tau = 5$ time units. Figure 2 shows the expected cost at time $t = 150$ versus the preventive threshold M. The data have been obtained using MC simulation for 50 values of M ranging from 0 to 10, with 4000 realizations in each point.

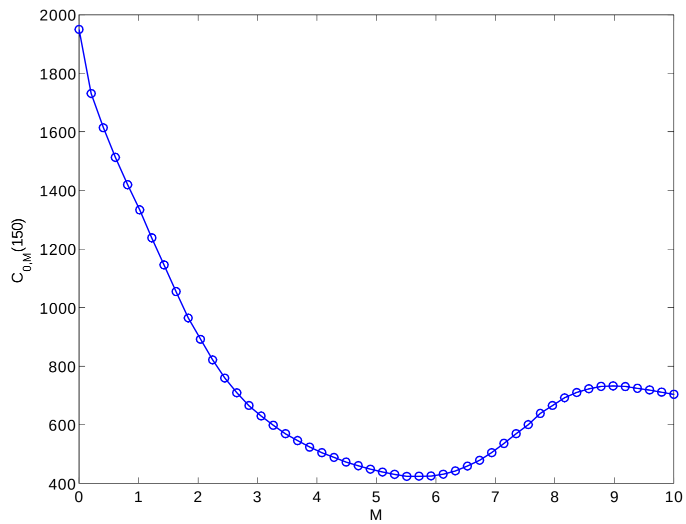

FIGURE 2: Expected cost at time $t = 150$ versus the preventive threshold M

Now, using Figure 2 we can find a value of M that minimizes $c_{0,M}(150)$, that is, find some M_{opt} such that

$$c_{0,M_{opt}}(150) = \inf \{c_{0,M}(150),\, 0 \leq M \leq 10\},$$

where $c_{0,M}(150)$ denotes the expected cost at time $t = 150$ for each value of M, starting from $Y_0 = 0$. By inspection, the expected cost $c_{0,M}(150)$ just presents a unique minimum and it is reached for $M_{opt} \simeq 5.5102$ with an expected cost of 423.9 monetary units.

Taking $\tau = 10$ time units, Figure 3 and 4 show values of the availability $A_{0,M}(75)$ and of the expected cost $c_{0,M}(75)$ at time $t = 75$ versus the preventive threshold M, respectively. These figures have been obtained using MC simulation for 100 values from 2 to 10, and 40000 realizations in each point. Based on Figure 3, we can see that the availability at time $t = 75$ reaches its minimum at $M^* \simeq 8.2222$, with $A_{0,M^*}(75) \simeq 0.5905$. Hence, for any value of M, the probability that the system is working at time $t = 75$ exceeds or is equal to 59.05%. Based on Figure 4, we can see that the cost function $c_{0,M}(75)$ reaches its minimum at $M^* \simeq 4.4242$, with $c_{0,M^*}(75) \simeq$ 316.0753 monetary units. Also, using both Figures 3 and 4, it is possible to find some optimal M^{**} minimizing the cost $c_{0,M}(75)$ under some availability constraint such as $A_{0,M}(75) \geq 0.99$ e.g.. This provides $M^{**} \simeq 3.8586$, with $c_{0,M^{**}}(75) \simeq 320.2977$ monetary units and $A_{0,M^{**}}(75) = 0.9905$. Symetrically, it also is possible to optimize the availability function under some cost constraint.

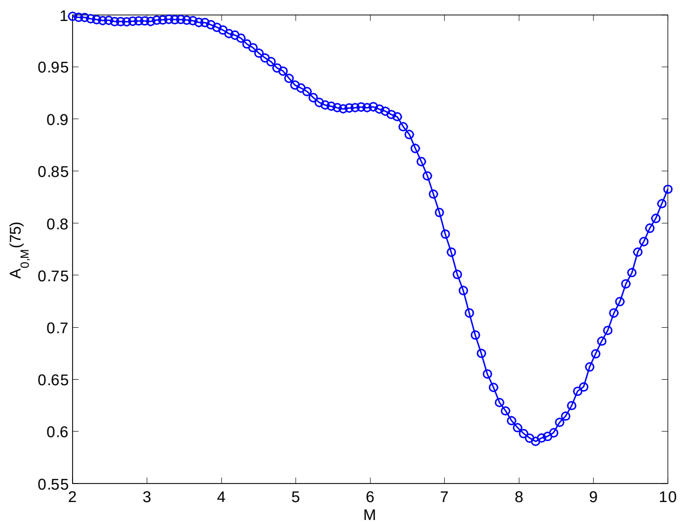

FIGURE 3: Availability at time $t = 75$ versus the preventive threshold M

Figure 5 shows the transient availability versus time for $\tau = 15$ and $M = 5$. This figure has been obtained using MC simulation for 350 values from 0 to 175 and 40000 realizations in each point. As can be observed in Figure 5, the availability function shows some alternating decreasing and increasing periods with respect of time, which can be explained by the following: at the beginning, there is no maintenance action and the availability function decreases with time t until the first maintenance action at time S_1 is more likely to have been performed, namely until the probability that $t > S_1$ becomes larger. Indeed, we observe that $A_{0,5}(t)$ decreases up to $t \simeq 22.5645$, to be compared with $\mathbb{E}(S_1) = \mathbb{E}(\sigma_M) + \tau \simeq 25.3111$ time units (and $\mathbb{E}(\sigma_L) \simeq 20.3912$ time units). After reaching its first minimum in $t \simeq 22.5645$, the availability function

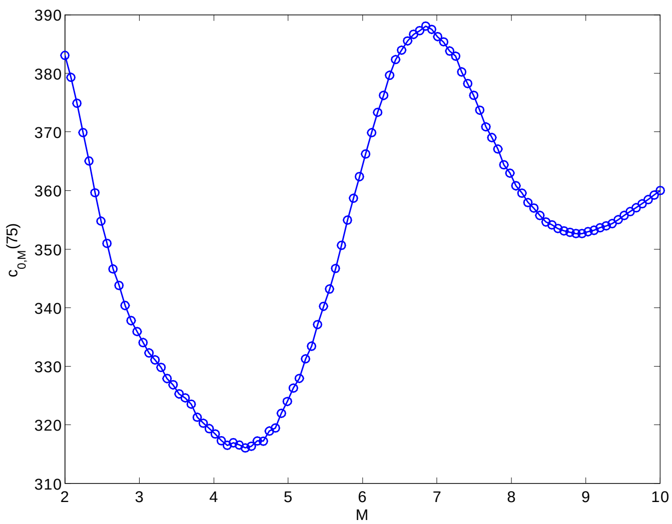

FIGURE 4: Expected cost at time $t = 75$ versus the preventive threshold M

increases along with the probability that a first maintence action has already been performed at time t. After a while, the probability that the system fails increases with the distance between t and the (nearly almost surely past) first maintenance action, which leads to a decreasing period, and a second minimum at $t \simeq 47.6361$, and so on. Note that the randomness of the maintenance times induces some attenuation in the decreasingness and increasingness over time.

Figure 6 finally shows the transient reliability versus the degradation level M at time $t = 50$ for $\tau = 15$. This figure has been performed using 50 points from 0 to 10 and 20000 realizations in each point. As we can check, the transient reliability is here decreasing against the preventive threshold M. This means that the shorter M is, the larger the reliability is. In this way, if the point is to maximize the reliability at time $t = 50$ with respect of M, the best is to take $M = 0$, namely perform periodic replacements. Though it seems challenging to prove it from Theorem 2, it seems to be coherent with intuition, because smaller M should involve more frequent replacements.

7. Conclusions and future extensions

In this work, the reliability of a system subject to a continuous degradation modelled as a gamma process with imperfect delay repair is analyzed. The functioning of the

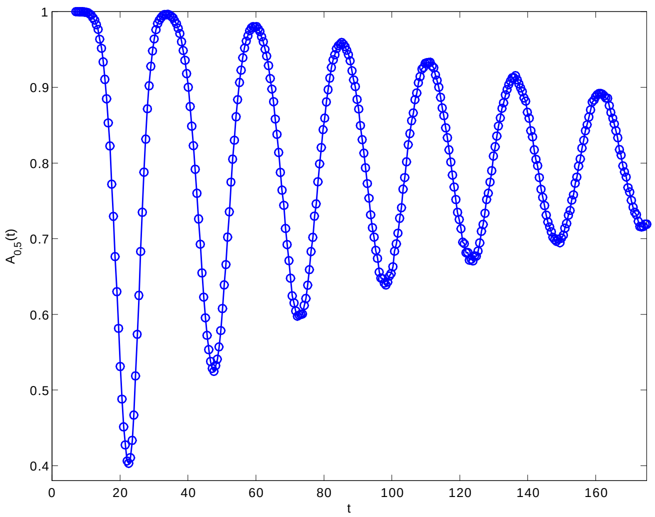

FIGURE 5: Transient availability versus time t given $M = 5$

system is described through a semi-regenerative process, obtaining that some transient reliability measures fulfill Markov renewal equations. Numerical examples of these reliability measures are showed. These numerical examples are obtained by Monte-Carlo simulation of the process due to the complexity of the Markov renewal equations, mainly caused by the overshoot of the gamma process and by the imperfect repair nature that reduces the system age. It would be interesting to compare this model (age-based repair) with a similar one where the imperfect repair would reduce the system degradation itself instead of the system age, leading to some kind of virtual degradation. This should be part of a future work.

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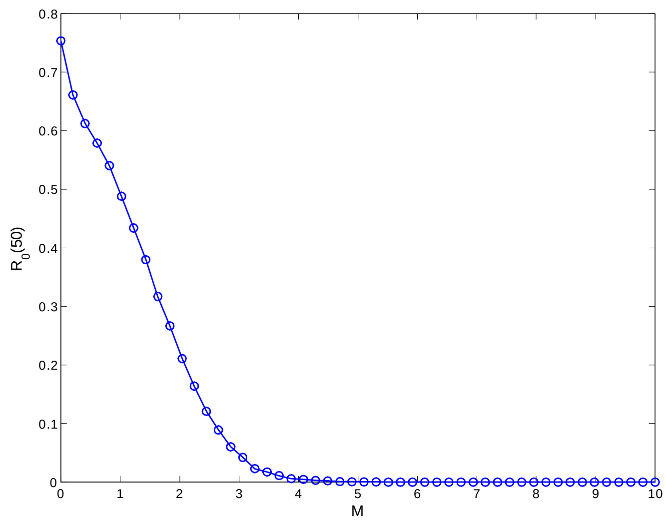

FIGURE 6: Transient reliability versus M at time $t = 50$

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